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Adel Alahmadi, Michel Planat, Patrick Solé

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# CHEBYSHEV'S BIAS AND GENERALIZED RIEMANN HYPOTHESIS

ADEL ALAMADHI\*, MICHEL PLANAT†, AND PATRICK SOLÉ†,\*

**ABSTRACT.** It is well known that  $\text{li}(x) > \pi(x)$  (i) up to the (very large) Skewes' number  $x_1 \sim 1.40 \times 10^{316}$  [1]. But, according to a Littlewood's theorem, there exist infinitely many  $x$  that violate the inequality, due to the specific distribution of non-trivial zeros  $\gamma$  of the Riemann zeta function  $\zeta(s)$ , encoded by the equation  $\text{li}(x) - \pi(x) \approx \frac{\sqrt{x}}{\log x} [1 + 2 \sum_{\gamma} \frac{\sin(\gamma \log x)}{\gamma}]$  (1). If Riemann hypothesis (RH) holds, (i) may be replaced by the equivalent statement  $\text{li}[\psi(x)] > \pi(x)$  (ii) due to Robin [2]. A statement similar to (i) was found by Chebyshev that  $\pi(x; 4, 3) - \pi(x; 4, 1) > 0$  (iii) holds for any  $x < 26861$  [3] (the notation  $\pi(x; k, l)$  means the number of primes up to  $x$  and congruent to  $l \pmod k$ ). The *Chebyshev's bias* (iii) is related to the generalized Riemann hypothesis (GRH) and occurs with a logarithmic density  $\approx 0.9959$  [3]. In this paper, we reformulate the Chebyshev's bias for a general modulus  $q$  as the inequality  $B(x; q, R) - B(x; q, N) > 0$  (iv), where  $B(x; k, l) = \text{li}[\phi(k) * \psi(x; k, l)] - \phi(k) * \pi(x; k, l)$  is a counting function introduced in Robin's paper [2] and  $R$  (resp.  $N$ ) is a quadratic residue modulo  $q$  (resp. a non-quadratic residue). We investigate numerically the case  $q = 4$  and a few prime moduli  $p$ . Then, we prove that (iv) is equivalent to GRH for the modulus  $q$ .

## 1. INTRODUCTION

In the following, we denote  $\pi(x)$  the prime counting function and  $\pi(x; q, a)$  the number of primes not exceeding  $x$  and congruent to  $a \pmod q$ . The asymptotic law for the distribution of primes is the prime number theorem  $\pi(x) \sim \frac{x}{\log x}$ . Correspondingly, one gets [4, eq. (14), p. 125]

$$(1.1) \quad \pi(x; q, a) \sim \frac{\pi(x)}{\phi(q)}$$

that is, one expects the same number of primes in each residue class  $a \pmod q$ , if  $(a, q) = 1$ . Chebyshev's bias is the observation that, contrarily to expectations,  $\pi(x; q, N) > \pi(x; q, R)$  most of the times, when  $N$  is a non-square modulo  $q$ , but  $R$  is.

Let us start with the bias

$$(1.2) \quad \delta(x, 4) := \pi(x; 4, 3) - \pi(x; 4, 1)$$

found between the number of primes in the non-quadratic residue class  $N = 3 \pmod 4$  and the number of primes in the quadratic one  $R = 1 \pmod 4$ . The values

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$\delta(10^n, 4)$ ,  $n \leq 1$ , form the increasing sequence

$$A091295 = \{1, 2, 7, 10, 25, 147, 218, 446, 551, 5960, \dots\}.$$

The bias is found to be negative in thin zones of size

$$\{2, 410, 15\,358, 41346, 42\,233\,786, 416\,889\,978, \dots\}$$

spread over the location of primes of maximum negative bias [5]

$$\{26861, 623\,681, 12\,366\,589, 951\,867\,937, 6\,345\,026\,833, 18\,699\,356\,321 \dots\}.$$

It has been proved that there are infinitely many sign changes in the Chebyshev's bias (1.2). This follows from the Littlewood's oscillation theorem [6, 7]

$$(1.3) \quad \delta(x, 4) := \Omega_{\pm} \left( \frac{x^{1/2}}{\log x} \log_3 x \right).$$

A useful measure of the Chebyshev's bias is the logarithmic density [3, 6, 8]

$$(1.4) \quad d(A) = \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{a \in A, a \leq x} \frac{1}{a}$$

for the positive  $\Delta^+$  and negative  $\Delta^-$  regions calculated as  $d(\Delta^+) = 0.9959$  and  $d(\Delta^-) = 0.0041$ .

In essence, Chebyshev's bias  $\delta(x, 4)$  is similar to the bias

$$(1.5) \quad \delta(x) := \text{Li}(x) - \pi(x).$$

It is known that  $\delta(x) > 0$  up to the (very large) Skewes' number  $x_1 \sim 1.40 \times 10^{316}$  but, according to Littlewood's theorem, there also are infinitely many sign changes of  $\delta(x)$  [7].

The reason why the asymmetry in (1.5) is so much pronounced is encoded in the following approximation of the bias [3, 9]<sup>1</sup>

$$(1.6) \quad \delta(x) \sim \frac{\sqrt{x}}{\log x} \left( 1 + 2 \sum_{\gamma} \frac{\sin(\gamma \log x + \alpha_{\gamma})}{\sqrt{1/4 + \gamma^2}} \right),$$

where  $\alpha_{\gamma} = \cot^{-1}(2\gamma)$  and  $\gamma$  is the imaginary part of the non-trivial zeros of the Riemann zeta function  $\zeta(s)$ . The smallest value of  $\gamma$  is quite large,  $\gamma_1 \sim 14.134$ , and leads to a large asymmetry in (1.5).

Under the assumption that the generalized Riemann hypothesis (GRH) holds that is, if the Dirichlet L-function with non trivial real character  $\kappa_4$

$$(1.7) \quad L(s, \kappa_4) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^s},$$

has all its non-trivial zeros located on the vertical axis  $\Re(s) = \frac{1}{2}$ , then the formula (1.6) also holds for the Chebyshev's bias  $\delta(x, 4)$ . The smallest non-trivial zero of  $L(s, \kappa_4)$  is at  $\gamma_1 \sim 6.02$ , a much smaller value than the one corresponding to  $\zeta(s)$ , so that the bias is also much smaller.

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<sup>1</sup>The bias may also be approached in a different way by relating it to the second order Landau-Ramanujan constant [10].

A second factor controls the aforementioned asymmetry of a  $L$ -function of real non-trivial character  $\kappa$ , it is the *variance* [11]

$$(1.8) \quad V(\kappa) = \sum_{\gamma > 0} \frac{2}{1/4 + \gamma^2}.$$

For the function  $\zeta(s)$  and  $L(s, \kappa_4)$  one gets  $V = 0.045$  and  $V = 0.155$ , respectively.

**Our main goal.** In a groundbreaking paper, Robin reformulated the unconditional bias (1.5) as a conditional one involving the second Chebyshev function  $\psi(x) = \sum_{p^k \leq x} \log p$

$$(1.9) \quad \text{The equality } \delta'(x) := \text{li}[\psi(x)] - \pi(x) > 0 \text{ is equivalent to RH.}$$

This statement is given as Corollary 1.2 in [12] and led the second and third author of the present work to derive a *good prime counting function*

$$(1.10) \quad \pi(x) = \sum_{n=1}^3 \mu(n) \text{li}[\psi(x)^{1/n}].$$

Here, we are interested in a similar method to *regularize* the Chebyshev's bias in a conditional way similar to (1.9). In [2], Robin introduced the function

$$(1.11) \quad B(x; q, a) = \text{li}[\phi(q)\psi(x; q, a)] - \phi(q)\pi(x; q, a),$$

that generalizes (1.9) and applies it to the residue class  $a \pmod q$ , with  $\psi(x, q, a)$  the generalized second Chebyshev's function. Under GRH he proved that [2, Lemma 2, p. 265]

$$(1.12) \quad B(x; q, a) = \Omega_{\pm} \left( \frac{\sqrt{x}}{\log^2 x} \right),$$

that is

$$(1.13) \quad \text{The inequality } B(x; q, a) > 0 \text{ is equivalent to GRH.}$$

For the Chebyshev's bias, we now need a proposition taking into account two residue classes such that  $a = N$  (a non-quadratic residue) and  $a = R$  (a quadratic one).

**Proposition 1.1.** Let  $B(x; q, a)$  be the Robin  $B$ -function defined in (1.11), and  $R$  (resp.  $N$ ) be a quadratic residue modulo  $q$  (resp. a non-quadratic residue), then the statement  $\delta'(x, q) := B(x; q, R) - B(x; q, N) > 0, \forall x$  (i), is equivalent to GRH for the modulus  $q$ .

The present paper deals about the numerical justification of proposition 1.1 in Sec. 2 and its proof in Sec. 3. The calculations are performed with the software Magma [13] available on a 96 MB segment of the cluster at the University of Franche-Comté.

## 2. THE REGULARIZED CHEBYSHEV'S BIAS

All over this section, we are interested in the prime champions of the Chebyshev's bias  $\delta(x, q)$  (as defined in (1.2) or (2.3), depending on the context). We separate the prime champions leading to a positive/negative bias. Thus, the  $n$ -th prime champion satisfies

$$(2.1) \quad \delta^{(\epsilon)}(x_n, q) = \epsilon n, \quad \epsilon = \pm 1.$$

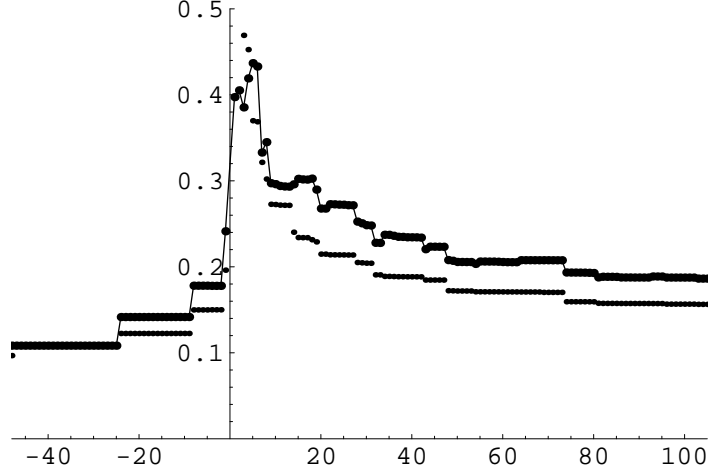


FIGURE 1. The normalized regularized bias  $\delta'(x, 4)/\sqrt{x}$  versus the Chebyshev's bias  $\delta(x, 4)$  at the prime champions of  $\delta(x, 4)$  (when  $\delta(x, 4) > 0$ ) and at the prime champions of  $-\delta(x, 4)$  (when  $\delta(x, 4) < 0$ ). The extremal prime champions in the plot are  $x = 359327$  (with  $\delta = 105$ ) and  $x = 951867937$  (with  $\delta = -48$ ). The curve is asymmetric around the vertical axis, a fact that reflects the asymmetry of the Chebyshev's bias. As explained in the text, a violation of GRH would imply a negative value of the regularized bias  $\delta'(x, 4)$ . The small dot curve corresponds to the fit of  $\delta'(x, 4)/\sqrt{x}$  by  $2/\log x$  calculated in Sec. 3.

We also introduce a new measure of the overall bias  $b(q)$ , dedicated to our plots, as follows

$$(2.2) \quad b(q) = \sum_{n, \epsilon} \frac{\delta^{(\epsilon)}(x_n, q)}{x_n}.$$

Indeed, smaller is the bias lower is the value of  $b(q)$ . Anticipating over the results presented below, Table 1 summarize the calculations.

TABLE 1. The new bias (2.2) (column 2) and the standard logarithmic density (1.4) (column 3).

modulus $q$	bias $b(q)$	log density $d(\Delta^+)$	first zero $\gamma_1$
4	0.7926	0.9959 [3]	14.134
11	0.1841	0.9167 [3]	0.2029
13	0.2803	0.9443 [3]	3.119
163	0.0809	0.55 [9]	2.477

**Chebyshev's bias for the modulus  $q = 4$ .** As explained in the introduction, our goal in this paper is to reexpress a standard Chebyshev's bias  $\delta(x, q)$  into a regularized one  $\delta'(x, q)$ , that is always positive under the condition that GRH holds. Indeed we do not discover any numerical violation of GRH and we always obtains a positive  $\delta'(x, q)$ . The asymmetry of Chebyshev's bias arises in the plot  $\delta$  vs  $\delta'$ , where the fall of the normalized bias  $\frac{\delta}{\sqrt{x}}$  is faster for negative values of  $\delta$  than for positive ones. Fig. 1 clarifies this effect for the historic modulus  $q = 4$ . We restricted our plot to the champions of the bias  $\delta$  and separated positive and negative champions.

**Chebyshev's bias for a prime modulus  $p$ .** For a prime modulus  $p$ , we define the bias so as to obtain an averaging over all differences  $\pi(x; p, N) - \pi(x; p, R)$ , where as above  $N$  and  $R$  denote a non-quadratic and a quadratic residue, respectively

$$(2.3) \quad \delta(x, p) = - \sum_a \left( \frac{a}{p} \right) \pi(x; p, a),$$

where  $\left( \frac{a}{p} \right)$  is the Legendre symbol. Correspondingly, we define the regularized bias as

$$(2.4) \quad \delta'(x, p) = \frac{1}{[p/2]} \sum_a \left( \frac{a}{p} \right) B(x; p, a).$$

**Proposition 2.1.** Let  $p$  be a selected prime modulus and  $\delta'(x, p)$  as in (2.4) then the statement  $\delta'(x, p) > 0, \forall x$ , is equivalent to GRH for the modulus  $p$ .

As mentioned in the introduction, the Chebyshev's bias is much influenced by the location of the first non-trivial zero of the function  $L(s, \kappa_q)$ ,  $\kappa_q$  being the real non-principal character modulo  $q$ . This is especially true for  $L(s, \kappa_{163})$  with its smaller non-trivial zero at  $\gamma \sim 0.2029$  [9]. The first negative values occur at  $\{15073, 15077, 15083, \dots\}$ .

Fig. 2 represents the Chebyshev's bias  $\delta'$  for the modulus  $q = 163$  versus the standard one  $\delta$  (thick dots). The asymmetry of the Chebyshev's bias is revealed at small values of  $|\delta|$  where the fit of the regularized bias by the curve  $2/\log x$  is not good (thin dots).

For the modulus  $q = 13$ , the imaginary part of the first zero is not especially small,  $\gamma_1 \sim 3.119$ , but the variance (1.8) is quite high,  $V(\kappa_{-13}) \sim 0.396$ . The first negative values of  $\delta(x, 13)$  at primes occur when  $\{2083, 2089, 10531, \dots\}$ . Fig. 3 represents the Chebyshev's bias  $\delta'$  for the modulus  $q = 13$  versus the standard one  $\delta$  (thick dots) as compared to the fit by  $2/\log x$  (thin dots).

Finally, for the modulus  $q = 11$ , the imaginary part of the first zero is quite small,  $\gamma_1 \sim 0.209$ , and the variance is high,  $V(\kappa_{-11}) \sim 0.507$ . In such a case, as shown in Fig. 4, the approximation of the regularized bias by  $2/\log x$  is good in the whole range of values of  $x$ .

### 3. PROOF OF PROPOSITION 1.1

For approaching the proposition 1.1 we reformulate it in a simpler way as

**Proposition 3.1.** One introduces the regularized counting function  $\pi'(x; q, l) := \pi(x; q, l) - \psi(x; q, l)/\log x$ . The statement  $\pi'(x; q, N) > \pi'(x; q, R), \forall x$  (ii), is equivalent to GRH for the modulus  $q$ .

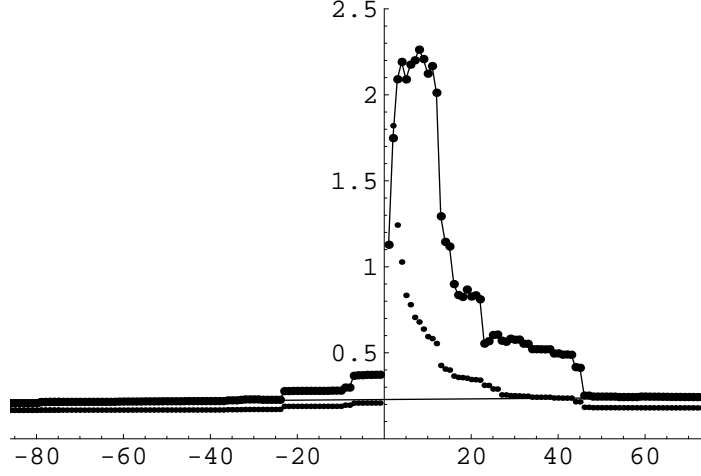


FIGURE 2. The normalized regularized bias  $\delta'(x, 163)/\sqrt{x}$  versus the Chebyshev's bias  $\delta(x, 163)$  at all the prime champions of  $|\delta(x, 163)|$  [from  $|\delta(x, 163)| > 74$  the bias is  $\delta(x, 163) < 0$  negative], superimposed to the curve at the prime champions of  $-\delta(x, 163)$  (when  $\delta(x, 163) < 0$ ). The extremal prime champions in the plot are  $x = 68491$  (with  $\delta = 74$ ) and  $x = 174637$  (with  $\delta = -86$ ). The asymmetry is still clearly visible in the range of small values of  $|\delta|$  but tends to disappear in the range of high values of  $|\delta|$ . The small dot curve corresponds to the fit of  $\delta'(x, 163)/\sqrt{x}$  by  $2/\log x$  calculated in Sec. 3.

*Proof.* First observe that proposition 1.1 follows from proposition 3.1. This is straightforward because according to [2, p. 260], the prime number theorem for arithmetic progressions leads to the approximation

$$(3.1) \quad \text{li}[\phi(q)\psi(x; q, l)] \sim \text{li}(x) + \frac{\phi(q)\psi(x; q, l) - x}{\log x}.$$

As a result

$$\begin{aligned} \delta'(x, q) &= B(x; q, R) - B(x; q, N) \\ &= \text{li}[\phi(q)\psi(x; q, R)] - \text{li}[\phi(q)\psi(x; q, N)] + \phi(q)\delta(x, q) \\ &\sim \phi(q)[\pi'(x; q, N) - \pi'(x; q, R)]. \end{aligned}$$

The asymptotic equivalence in (3.1) holds up to the error term [2, p. 260]  $O(\frac{R(x)}{x \log x})$ , with

$$R(x) = \min \left( x^{\theta_q} \log^2 x, x e^{-a\sqrt{\log x}} \right), \quad a > 0,$$

$$\theta_q = \max_{\kappa \pmod q} (\sup \Re(\rho), \quad \rho \text{ a zero of } L(s, \kappa)).$$

Let us now look at the statement  $\text{GRH} \Rightarrow (i)$ . Following [3, p 178-179], one has

$$\psi(x; q, a) = \frac{1}{\phi(q)} \sum_{\kappa \pmod q} \bar{\kappa}(a) \psi(x, \kappa)$$

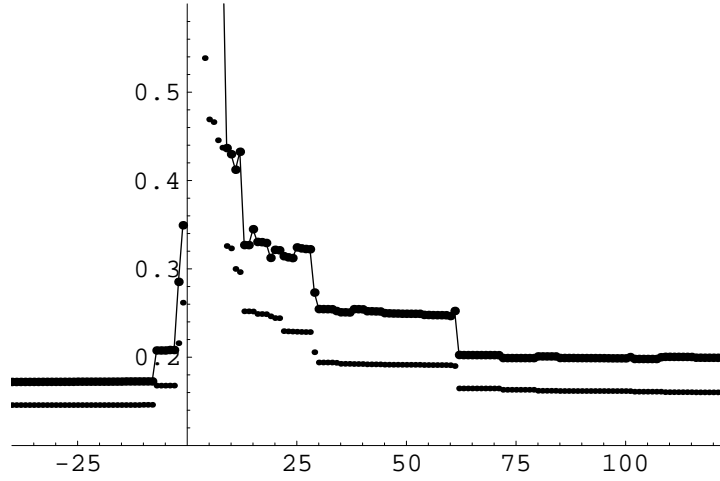


FIGURE 3. The normalized regularized bias  $\delta'(x, 13)/\sqrt{x}$  versus the Chebyshev's bias  $\delta(x, 13)$  at the prime champions of  $\delta(x, 13)$  (when  $\delta(x, 13) > 0$ ), and the curve at the prime champions of  $-\delta(x, 13)$  (when  $\delta(x, 13) < 0$ ). The extremal prime champions in the plot are  $x = 263881$  (with  $\delta = 123$ ) and  $x = 905761$  (with  $\delta = -40$ ). The small dot curve corresponds to the fit of  $\delta'(x, 13)/\sqrt{x}$  by  $2/\log x$  calculated in Sec. 3.

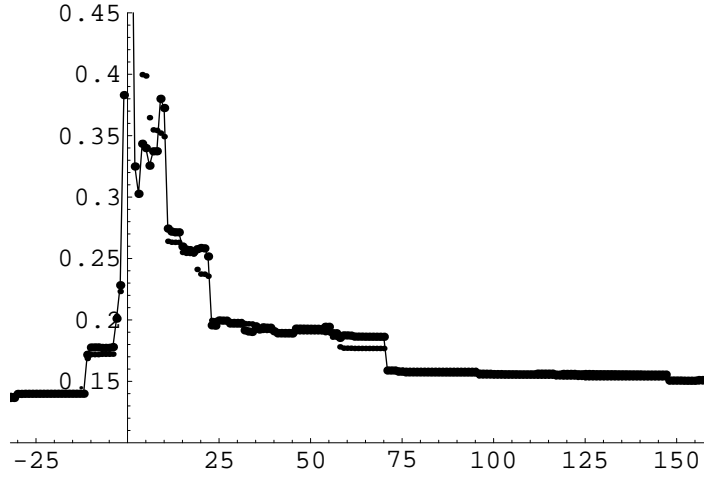


FIGURE 4. The normalized regularized bias  $\delta'(x, 11)/\sqrt{x}$  versus the Chebyshev's bias  $\delta(x, 11)$  at the prime champions of  $\delta(x, 11)$  (when  $\delta(x, 11) > 0$ ), and the curve at the prime champions of  $-\delta(x, 11)$  (when  $\delta(x, 11) < 0$ ). The extremal prime champions in the plot are  $x = 638567$  (with  $\delta = 158$ ) and  $x = 1867321$  (with  $\delta = -32$ ). The small dot curve corresponds to the (very good) fit of  $\delta'(x, 11)/\sqrt{x}$  by  $2/\log x$  calculated in Sec. 3.



and under GRH

$$\pi(x; q, a) = \frac{\pi(x)}{\phi(q)} - \frac{c(q, a)}{\phi(q)} \frac{\sqrt{x}}{\log x} + \frac{1}{\phi(q) \log x} \sum_{\kappa \neq \kappa_0} \bar{\kappa}(a) \psi(x, \kappa) + O\left(\frac{\sqrt{x}}{\log^2 x}\right),$$

where  $\kappa_0$  is the principal character modulo  $q$  and

$$c(q, a) = -1 + \#\{1 \leq b \leq q : b^2 = a \pmod{q}\}$$

for coprimes integers  $a$  and  $q$ . Note that for an odd prime  $q = p$ , one has  $c(p, a) = \left(\frac{a}{p}\right)$ .

Thus, under GRH

$$(3.2) \quad \begin{aligned} \pi(x; q, N) - \pi(x; q, R) &= \frac{1}{\phi(q) \log x} [\sqrt{x}(c(q, R) - c(q, N)) \\ &+ \sum_{\kappa \pmod{q}} (\bar{\kappa}(N) - \bar{\kappa}(R)) \psi(x, \kappa) + O\left(\left(\frac{\sqrt{x}}{\log^2 x}\right)\right]. \end{aligned}$$

The sum could be taken over all characters because  $\bar{\kappa}_0(N) = \bar{\kappa}_0(R)$ . In addition, we have

$$(3.3) \quad \psi(x; q, N) - \psi(x; q, R) = \frac{1}{\phi(q)} \sum_{\kappa \pmod{q}} [\bar{\kappa}(N) - \bar{\kappa}(R)] \psi(x, \kappa).$$

Using (3.2) and (3.3) the regularized bias reads

$$(3.4) \quad \begin{aligned} \delta'(x, q) &\sim \pi'(x; q, N) - \pi'(x; q, R) \\ &= \frac{\sqrt{x}}{\log x} [c(q, R) - c(q, N)] + O\left(\frac{\sqrt{x}}{\log^2 x}\right). \end{aligned}$$

For the modulus  $q = 4$ , we have  $c(q, 1) = -1 + 2 = 1$  and  $c(q, 3) = -1$  so that  $\delta'(x, 4) = \frac{2\sqrt{x}}{\log x}$ . The same result is obtained for a prime modulus  $q = p$  since  $c(p, N) = -1$  and  $c(p, R) = c(p, 1) = \left(\frac{1}{p}\right) = 1$ .

This finalizes the proof that under GRH, one has the inequality  $\pi'(x; q, N) > \pi'(x; q, R)$ .

If GRH does not hold, then using [2, lemma 2], one has

$$B(x; q, a) = \Omega_{\pm}(x^{\xi}) \text{ for any } \xi < \theta_q.$$

Applying this asymptotic result to the residue classes  $a = R$  and  $a = N$ , there exist infinitely many values  $x = x_1$  and  $x = x_2$  satisfying

$$B(x_1; q, R) < -x_1^{\xi} \text{ and } B(x_2; q, N) > x_2^{\xi} \text{ for any } \xi < \theta_q,$$

so that one obtains

$$(3.5) \quad B(x_1; q, R) - B(x_2; q, N) < -x_1^{\xi} - x_2^{\xi} < 0.$$

Selecting a pair  $(x_1, x_2)$  either

$$B(x_1; q, R) > B(x_2; q, R)$$

so that  $B(x_2; q, R) - B(x_2; q, N) < 0$  and (i) is violated at  $x_2$ , or

$$(3.6) \quad B(x_1; q, R) < B(x_2; q, R).$$

In the last case, either  $B(x_1; q, N) > B(x_2; q, N)$ , so that  $B(x_1; q, R) - B(x_1; q, N) < 0$  and the inequality (i) is violated at  $x_1$ , or simultaneously

$$B(x_1; q, N) < B(x_2; q, N) \text{ and } B(x_1; q, R) < B(x_2; q, R),$$

which implies (3.5) and the violation of (i) at  $x = x_1 = x_2$ .

To finalize the proof of 3.1, and simultaneously that of 1.1, one makes use of the asymptotic equivalence of (i) and (ii), that is if GRH is true  $\Rightarrow$  (ii)  $\Rightarrow$  (i), and if GRH is wrong, (i) may be violated and (ii) as well.

Then, proposition 2.1 also follows as a straightforward consequence of proposition 1.1. □

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#### 4. SUMMARY

We have found that the asymmetry in the prime counting function  $\pi(x; q, a)$  between the quadratic residues  $a = R$  and the non-quadratic residues  $a = N$  for the modulus  $q$  can be encoded in the function  $B(x; q, a)$  [defined in (1.11)] introduced by Robin in the context of GRH [2], or into the regularized prime counting function  $\pi'(x; q, a)$ , as in Proposition 3.1. The bias in  $\pi'$  reflects the bias in  $\pi$  conditionally under GRH for the modulus  $q$ . Our conjecture has been initiated by detailed computer calculations presented in Sec. 2 and proved in Sec. 3. Further work could follow the work about the connection of  $\pi$ , and thus of  $\pi'$ , to the sum of squares function  $r_2(n)$  [10].

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\*MECAA, KING ABDULAZIZ UNIVERSITY, JEDDAH, SAUDI ARABIA.

†INSTITUT FEMTO-ST, CNRS, 32 AVENUE DE L'OBSERVATOIRE, F-25044 BESANÇON, FRANCE.  
E-mail address: michel.planat@femto-st.fr

‡TELECOM PARISTECH, 46 RUE BARRAULT, 75634 PARIS CEDEX 13, FRANCE.  
E-mail address: sole@telecom-paristech.fr